Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 5, pp. 613–621, 2004

GALILEAN-INVARIANT AND THERMODYNAMICALLY CONSISTENT MODEL OF A COMPOSITE ISOTROPIC MEDIUM

S. K. Godunov

UDC 517:9:539:3

The hydrodynamics of a multiphase media or a mixture is formalized in the form of a hyperbolic system taking into account chemical reactions. Additional conditions consistent with the system are found for the solutions to ensure conservation of energy and momentum.

Key words: *reproducing potential, hyperbolicity, dissipation, entropy.*

Introduction. The wide use of various mathematical models for continuous media in commercial calculations and analytical studies has led to the problem of formalizing the laws of thermodynamics for the systems of equations governing the behavior of solutions of such models.

Examples of such formalizations are described in [1–5]. The present paper continues collecting systems of Galilean-invariant equations that can be defined concretely by specifying one reproducing potential L expressed in terms of unknown functions u_i, q_j, $\ldots$, T.

Section 1 deals with the equations of gas dynamics in which the gas is a mixture of chemical substances reacting with each other. In essence, here the question is one more possible formulation based on the well-known description of reactions using the law of mass action. As unknown functions we have to choose, instead of ordinary Gibbs potentials, some of their combinations that are more suitable in using Euler coordinates.

Equations from Sec. 2 can be to regarded as a complicated version of the equations from Sec. 1. The medium in this case is also treated as a mixture of reactants but the reactants are assumed to be broken up into small-size meshes adjoining each other and the phase transitions between them are modeled by equations of the same type as the reactions in Sec. 1. Although the meshes are treated as microscopic details constituting a macroscopic medium described by our equations, the presence of these details is a reason for which it is necessary to take into account the inhomogeneity of the velocity fields inside the meshes (it can be assumed, for example, that the motion of the mesh boundaries causes the added-mass effect in them, resulting in a difference between the velocity near the boundaries and the velocity at the centers of the meshes or it can be assumed that the bubble meshes move relative to the matrix material in which they are impregnated). To take into account inhomogeneity of this kind, we use, in addition to the momentum vector with components ρu_i averaged over neighboring meshes, one more vector characteristic with components $v_i^{(j)}$ that has the dimension of momentum and characterizes the inhomogeneity of the velocity field. (In conversion to a new coordinate system moving at a constant velocity relative to the old one, the components u_i are replaced by $u_i + U_i$ and $v_i^{(j)}$ do not change.)

Our model has evolved as a natural formal modification of the equations used by physicists for superfluid liquids (see [6]). Its development was motivated by theoretical papers [7–9] on modeling a liquid with impregnated bubbles. These papers pointed out the necessity of introducing two velocity fields to take into account the liquid masses added to the bubbles. Unfortunately, we were unable to use the results of [7–9] in the derivation of our equations describing a medium whose elements contain averages over a large number of bubbles. This is partly due to the fact that in the cited papers, the liquid is incompressible whereas we take into account its compressibility, and partly because it is unclear how to complete these studies with allowance for the statistics of bubble oscillations, which would enable accounting for their effect on the medium in a thermodynamic way.

Sobolev Institute of Mathematics, Novosibirsk 630090; godunov@math.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 5, pp. 3–12, September–October, 2004. Original article submitted February 4, 2004.

We do not attempt to derive the equations studied in Sec. 2 and confine ourselves to the above brief explanation of the reasons for the choice of the parameters describing the medium. At the same time, we substantiate the correctness (in a local sense) of the studied equations in the nondissipative version in great detail. It turns out that our system is symmetric hyperbolic and it is possible to distinguish a class of its solutions for which the physically meaningful conservation laws hold. The inclusion of reactions and dissipative terms does not violate these conservation laws but modifies them, transforming one of them to the law of entropy increase.

All these facts, in essence, serve to justify the reasonableness of the choice of the proposed system from nontrivial theoretical considerations. We would like to attract the attention of researchers dealing with multiphase media to the discussion of the model described here. Of course, this model cannot be considered complete but we hope that its analysis may be useful.

1. Gas Dynamics in a Nearly Equilibrium Chemical Medium. The state of an element of the media considered in this section is described by the pressure P, temperature T, internal energy E, entropy S, and volume V per unit mass. The density ρ is related to the specific volume by equality $\rho = 1/V$. The elemental composition is described by the numbers N_j, i.e., the number of gram molecules of the jth substance in unit mass (in one gram). In this case,

$$\rho = \sum_j \rho N_j, \qquad \sum_j N_j = 1.$$

The velocity components of the center of gravity of the element are denoted by u_i.

The so-called thermodynamic potential of the medium

$$\Phi = E - TS + PV$$

is defined by the equation of state

$$\Phi = \Phi(T, P, N_1, N_2, \ldots).$$

The thermodynamic identity

$$d\Phi = -S\,dT + V\,dP + \mu_k\,dN_k$$

relates Φ to the parameters S and V and the so-called Gibbs chemical potentials μ_j. This identity implies

$$dP = \rho\,d(\Phi - \mu_k N_k - u_i u_i/2) + \rho u_i\,du_i + \rho S\,dT + \rho N_j\,d\mu_j$$

$$= \Big(\sum_j \rho N_j\Big)\,d(\Phi - \mu_k N_k - u_i u_i/2) + \rho u_i\,du_i + \rho S\,dT + \rho N_j\,d\mu_j$$

$$= \rho N_j\,d(\mu_j + \Phi - \mu_k N_k - u_i u_i/2) + \rho u_i\,du_i + \rho S\,dT.$$

Introducing the notation

$$q_j = \mu_j + (\Phi - \mu_k N_k) - u_i u_i/2, \qquad L = P,$$

we obtain the thermodynamic relation

$$dL = \rho N_j\,dq_j + \rho u_i\,du_i + \rho S\,dT$$

for the reproducing potential L, which is used to describe the system of equations governing the motion of the medium and processes in it. This reproducing potential is given by the equation of state

$$L = L(q_1, q_2, \ldots, u_1, u_2, u_3, T) = \Lambda(q_1 + u_i u_i/2, q_2 + u_i u_i/2, \ldots, T).$$

In this case,

$$\rho N_j = L_{q_j}, \qquad \rho u_i = L_{u_i}, \qquad \rho S = L_T, \qquad \rho = \sum_j L_{q_j},$$

and the total energy $\mathcal{E}$ (internal and kinetic) of unit volume is defined by the formula

$$\rho(E + u_i u_i/2) = \mathcal{E} = q_j L_{q_j} + u_i L_{u_i} + T L_T - L.$$

In other words, $\mathcal{E}$ is this Legendre transform of the potential L with respect to its arguments q_j, u_i, and T.

If the motion of the medium is not accompanied by chemical reactions, it is described by the gas-dynamic equations

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial\,(\rho u_i u_k + P)}{\partial x_k} = 0, \qquad \frac{\partial \rho N_j}{\partial t} + \frac{\partial\,(u_k \rho N_j)}{\partial x_k} = 0, \qquad \frac{\partial \rho S}{\partial t} + \frac{\partial\,(u_k \rho S)}{\partial x_k} = 0,$$

in which the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial\,(u_k \rho)}{\partial x_k} = 0$$

is replaced by the mass conservation laws

$$\frac{\partial \rho N_j}{\partial t} + \frac{\partial\,(u_k \rho N_j)}{\partial x_k} = 0$$

for each component. The sum of equalities expressing these laws coincides with the continuity equation. In the notation introduced above, the gas-dynamic equations are written as

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,(u_k L)_{u_i}}{\partial x_k} = 0, \qquad \frac{\partial L_{q_j}}{\partial t} + \frac{\partial\,(u_k L)_{q_j}}{\partial x_k} = 0, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L)_T}{\partial x_k} = 0.$$

Linear combination of the above equalities with the coefficients u_i, q_j, and T based on the identities

$$u_i\, dL_{u_i} + q_j\, dL_{q_j} + T\, dL_T = d\mathcal{E},$$

$$u_i\, d(u_k L)_{u_i} + q_j\, d(u_k L)_{q_j} + T\, d(u_k L)_T = d[u_k(\mathcal{E} + L)]$$

leads to the energy conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial\,[u_k(\mathcal{E} + L)]}{\partial x_k} = 0.$$

The reactions proceeding in a medium with a nearly equilibrium chemical state can be described using the law of mass action (see, for example, [10, 11]):

$$\frac{\partial N_j}{\partial t} + u_k\,\frac{\partial N_j}{\partial x_k} \equiv \frac{dN_j}{dt} = -\sum_s \nu_j^{(s)} \Big(\frac{\nu_l^{(s)} \mu_l}{\tau_s}\Big).$$

The stoichiometric coefficient $\nu_j^{(s)}$ is proportional to the number of gram moles of the jth reactant consumed (if $\nu_j^{(s)} > 0$) or reproduced (if $\nu_j^{(s)} < 0$) in the reaction with number s; μ_k is the chemical potential of the kth reactant. By τ_s we denote a certain characteristic time of the sth reaction. (The reaction rate is proportional to $1/\tau_s$.) From the mass conservation law, it follows that the stoichiometric coefficients for each fixed s should be lined by the relation

$$\sum_k \nu_k^{(s)} = 0.$$

Therefore,

$$\nu_l^{(s)} \mu_l = \nu_l^{(s)} [\mu_l + (\Phi - \mu_j N_j - u_i u_i/2)] = \nu_l^{(s)} q_l,$$

and the reaction equations are rewritten as

$$\frac{dN_j}{dt} = -\sum_s \nu_j^{(s)} \Big(\frac{\nu_l^{(s)} q_l}{\tau_s}\Big), \qquad \frac{\partial \rho N_j}{\partial t} + \frac{\partial\,(u_k \rho N_j)}{\partial x_k} = -\rho \sum_s \nu_j^{(s)} \Big(\frac{\nu_l^{(s)} q_l}{\tau_s}\Big).$$

The second of these equalities is obtained from the first using the continuity equation and, in the employed variables, it has the form

$$\frac{\partial L_{q_j}}{\partial t} + \frac{\partial\,(u_k L)_{q_j}}{\partial x_k} = -\Big(\sum_l L_{q_l}\Big) \sum_s \nu_j^{(s)} \Big(\frac{\nu_l^{(s)} q_l}{\tau_s}\Big).$$

It is necessary to introduce a nonzero right side into the entropy equation:

$$\frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L_T)}{\partial x_k} = \Big(\frac{1}{T}\sum_l L_{q_l}\Big)\sum_s \frac{(\nu_l^{(s)} q_l)^2}{\tau_s} \geqslant 0.$$

As a result, the gas-dynamic equations taking into account chemical reactions become

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,(u_k L)_{u_i}}{\partial x_k} = 0,$$

$$\frac{\partial L_{q_j}}{\partial t} + \frac{\partial\,(u_k L)_{q_j}}{\partial x_k} = -\Big(\sum_l L_{q_l}\Big)\nu_j^{(s)}\Big(\frac{\nu_l^{(s)} q_l}{\tau_s}\Big),$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L)_T}{\partial x_k} = \Big(\frac{1}{T}\sum_l L_{q_l}\Big)\sum_s \frac{(\nu_l^{(s)} q_l)^2}{\tau_s},$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial\,[u_k(\mathcal{E} + L)]}{\partial x_k} = 0.$$

This system includes the last equality — the energy conservation law. As in the case of no reactions, this equality is obtained from the previous ones by multiplying into the corresponding factors u_i, q_i, and T. In order to obtain a zero right side in the energy Equation in this case, we had to define the right side in the entropy equation in a special manner. This ensured satisfaction of the entropy increase law.

Next, the equations are modified by including in them terms describing viscosity, thermal conductivity, diffusion, and thermal diffusion. The thermal conductivity is denoted by K, and the matrix diffusion coefficients and the vector thermal diffusivities are denoted by D_{jm} and B_j, respectively. The viscous stress tensor σ_{ik}, as usual, is expressed in terms of the derivatives of the velocity and the nonnegative viscosities α and β:

$$\sigma_{ik} = \frac{\alpha}{2}\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big) + \beta\delta_{ik}\,\frac{\partial u_m}{\partial x_m}.$$

In this case, the power of the work of viscous forces in unit volume is nonnegative:

$$\sigma_{ik}\,\frac{\partial u_i}{\partial x_k} = \alpha\sum_{ik}\Big[\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big)\Big]^2 + \beta\Big(\frac{\partial u_m}{\partial x_m}\Big)^2 \geqslant 0.$$

We write the final equations, including the energy conservation law in them without additional explanations in the hope that this will not cause difficulties for the close reader:

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,(u_k L)_{u_i}}{\partial x_k} = \frac{\partial}{\partial x_k}\,\sigma_{ik},$$

$$\frac{\partial L_{q_j}}{\partial t} + \frac{\partial\,(u_k L)_{q_j}}{\partial x_k} = -\Big(\sum_l L_{q_l}\Big)\nu_j^{(s)}\Big(\frac{(\nu_l^{(s)} q_l)^2}{\tau_s}\Big) + \frac{\partial}{\partial x_k}\Big[D_{jm}\,\frac{\partial q_m}{\partial x_k} + B_j\,\frac{\partial T}{\partial x_k}\Big],$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L)_T}{\partial x_k} = \frac{\partial}{\partial x_k}\Big(\frac{K}{T}\,\frac{\partial T}{\partial x_k}\Big)$$

$$+\,\frac{1}{T}\Big[\frac{\partial u_i}{\partial x_k}\,\sigma_{ik}\Big(\sum_l L_{q_l}\Big)\Big(\frac{\nu_l^{(s)} q_l}{\tau_s}\Big)^2 + \frac{\partial q_j}{\partial x_k}\Big(D_{jm}\,\frac{\partial q_m}{\partial x_k} + B_j\,\frac{\partial T}{\partial x_k}\Big) + K\,\frac{\partial T}{\partial x_k}\,\frac{\partial T}{\partial x_k}\Big],$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_k}\Big[u_k(\mathcal{E} + L) - u_i\sigma_{ik} - q_j\Big(D_{jm}\,\frac{\partial q_m}{\partial x_k} + B_j\,\frac{\partial T}{\partial x_k}\Big) - K\,\frac{\partial T}{\partial x_k}\Big] = 0.$$

The quadratic form of the derivatives

$$\frac{\partial q_j}{\partial x_k}\,D_{jm}\,\frac{\partial q_m}{\partial x_k} + \frac{\partial q_j}{\partial x_k}\,B_j\,\frac{\partial T}{\partial x_k} + K\,\frac{\partial T}{\partial x_k}\,\frac{\partial T}{\partial x_k} \geqslant 0$$

should be assumed to be nonnegative definite to ensure a nondecrease of entropy in adiabatically insulated volumes.

2. Accounting for Interaction between the Constituents (Phases) of the Medium. Before considering new thermodynamically consistent models, we give a formal description of a somewhat modified form of the mathematical model of a superfluid liquid (see[9]). This form was proposed by Romensky to model a bubble medium. Similar equations have been used in modeling geophysical processes [12–15]. The basic equations are not divergent:

$$\begin{gathered}
\frac{\partial L_{u_i}}{\partial t}+\frac{\partial\,(u_k L)_{u_i}}{\partial x_k}+L_{v_i}\frac{\partial v_k}{\partial x_k}-L_{v_k}\frac{\partial v_k}{\partial x_i}=0,\\
\frac{\partial L_{v_i}}{\partial t}+\frac{\partial\,(u_k L)_{v_i}}{\partial x_k}-L_{v_i}\frac{\partial u_k}{\partial x_k}+L_{v_k}\frac{\partial u_k}{\partial x_i}+\frac{\partial q}{\partial x_i}=0,\\
\frac{\partial L_q}{\partial t}+\frac{\partial\,(u_k L)_q}{\partial x_k}+\frac{\partial v_k}{\partial x_k}=0,\\
\frac{\partial L_T}{\partial t}+\frac{\partial\,(u_k L)_T}{\partial x_k}=0.
\end{gathered}\tag{2.1}$$

At the same time, if the reproducing potential

$$L=L(q,u_1,u_2,u_3,v_1,v_2,v_3,T)$$

is a convex function of its arguments and if

$$L=\Lambda(q+u_iu_i/2,v_iv_i,T),$$

then these equations are Galilean-invariant and can be rewritten in quasilinear form as symmetric, Friedrichs-hyperbolic equations. Therefore, if the initial data for the system in question are smooth enough, then the system has a unique fairly smooth solution in a certain finite time interval. For each solution, the trajectories $dx_k/dt=u_k$ are characteristics, along which the following relations hold:

$$\frac{d\Omega}{dt}\equiv\frac{\partial\Omega_i}{\partial t}+u_k\frac{\partial\Omega_i}{\partial x_k}=\Omega_k\frac{\partial u_i}{\partial x_k}-\Omega_i\frac{\partial u_k}{\partial x_i}.\tag{2.2}$$

Here
$$\Omega_1=\frac{\partial L_{v_2}}{\partial x_3}-\frac{\partial L_{v_3}}{\partial x_2},\qquad \Omega_2=\frac{\partial L_{v_3}}{\partial x_1}-\frac{\partial L_{v_1}}{\partial x_3},\qquad \Omega_3=\frac{\partial L_{v_1}}{\partial x_2}-\frac{\partial L_{v_2}}{\partial x_1}.$$

Relations (2.2) are ordinary homogeneous linear differential equations, for which the uniqueness theorem states that if at the initial time, $\Omega_k=0$ ($k=1,2,3$), then these equalities hold true in the future. In other words, Eqs. (2.1) are consistent with the additional equations

$$\Omega_i=\varepsilon_{ijk}\frac{\partial L_{v_j}}{\partial x_k}=0.\tag{2.3}$$

For the solutions subject to (2.3), Eqs. (2.1) are equivalent to the conservation laws

$$\begin{gathered}
\frac{\partial L_{u_i}}{\partial t}+\frac{\partial}{\partial x_k}\left[(u_kL)_{u_i}+v_kL_{v_i}-\delta_{ik}v_rL_{v_r}\right]=0,\\
\frac{\partial L_{v_i}}{\partial t}+\frac{\partial}{\partial x_i}\left[(u_kL)_{v_k}+q\right]=0,\qquad \frac{\partial L_q}{\partial t}+\frac{\partial}{\partial x_k}\left[(u_kL)_q+v_k\right]=0,\\
\frac{\partial L_T}{\partial t}+\frac{\partial}{\partial x_k}(u_kL)_T=0,\\
\frac{\partial \mathcal{E}}{\partial t}+\frac{\partial}{\partial x_k}\left[u_k(\mathcal{E}+L-v_iL_{v_i})+v_k(u_iL_{v_i}+q)\right]=0,
\end{gathered}$$

whose number is greater by unity than the number of unknown functions equal to the number of the basic equations (2.1).

As in Sec. 1, $\mathcal{E}$ denotes the Legendre transform of the reproducing potential with respect to its arguments:

$$\mathcal{E}=qL_q+u_iL_{u_i}+v_iL_{v_i}+TL_T-L.$$

All these properties of system (2.1) are described in detail in [5, 9] and in the Appendix to [3].

Our goal is to propose a multiphase model that generalizes system (2.1). Instead of the unknown function q, as in Sec. 1, we use variables q_1, q_2, $\dots$, q_j, $\dots$, each of which is assigned a corresponding vector variable with components $v_i^{(j)}$ $(i = 1, 2, 3)$. Using the reproducing potential

$$\begin{aligned} L &= L(q_1, q_2, \dots, q_j, \dots, u_1, u_2, u_3, T, v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots, v_1^{(j)}, v_2^{(j)}, v_3^{(j)}, \dots, T) \\ &= \Lambda(q_1 + u_i u_i/2, q_2 + u_i u_i/2, \dots, q_j + u_i u_i/2, v_i^{(1)} v_i^{(1)}, \dots, v_i^{(j)} v_i^{(j)}, \dots, v_i^{(j)} v_i^{(k)}, \dots, T) \end{aligned} \tag{2.4}$$

we construct a system composed of the recurring second and third lines of (2.1) with particular q_j and $v_i^{(j)}$ in each line. The first line in (2.1) is supplemented by new terms $L_{v_i^{(j)}}\, \partial v_k^{(j)}/\partial x_k - L_{v_k^{(j)}}\, \partial v_k^{(j)}/\partial x_i$, and the last line remains unchanged:

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,(u_k L)_{u_i}}{\partial x_k} + \sum_j \Big(L_{v_i^{(j)}} \frac{\partial v_k^{(j)}}{\partial x_k} - L_{v_k^{(j)}} \frac{\partial v_k^{(j)}}{\partial x_i}\Big) = 0,$$

$$\frac{\partial L_{v_i^{(j)}}}{\partial t} + \frac{\partial\,(u_k L)_{v_i^{(j)}}}{\partial x_k} - \Big(L_{v_i^{(j)}} \frac{\partial u_k}{\partial x_k} - L_{v_k^{(j)}} \frac{\partial u_k}{\partial x_i}\Big) + \frac{\partial q_i}{\partial x_i} = 0,$$

$$\frac{\partial L_{q_j}}{\partial t} + \frac{\partial\,[(u_k L)_{q_j} + v_k^{(j)}]}{\partial x_k} = 0, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L)_T}{\partial x_k} = 0.$$

This system will now be further modified. By analogy with Sec. 1, we model the reactions in question, discarding the zero right terms in the last two lines:

$$\begin{aligned} &\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,(u_k L)_{u_i}}{\partial x_k} + \sum_j \Big(L_{v_i^{(j)}} \frac{\partial v_k^{(j)}}{\partial x_k} - L_{v_k^{(j)}} \frac{\partial v_k^{(j)}}{\partial x_i}\Big) = 0, \\ &\frac{\partial L_{v_i^{(j)}}}{\partial t} + \frac{\partial\,(u_k L)_{v_i^{(j)}}}{\partial x_k} - \Big(L_{v_i^{(j)}} \frac{\partial u_k}{\partial x_k} - L_{v_k^{(j)}} \frac{\partial u_k}{\partial x_i}\Big) + \frac{\partial q_j}{\partial x_i} = 0, \\ &\frac{\partial L_{q_j}}{\partial t} + \frac{\partial\,[(u_k L)_{q_j} + v_k^{(j)}]}{\partial x_k} = -\Big(\sum_l L_{q_l}\Big) \sum_s \nu_j^{(s)} \Big(\frac{\nu_l^{(s)} q_l^{(s)}}{\tau_s}\Big), \\ &\frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L)_T}{\partial x_k} = \Big(\frac{1}{T} \sum_l L_{q_l}\Big) \sum_s \frac{(\nu_l^{(s)} q_l^{(s)})^2}{\tau_s}. \end{aligned} \tag{2.5}$$

Using the same line of reasoning as for system (2.1), it can be proved that system (2.5) in quasilinear form is symmetric hyperbolic and Galilean-invariant under the assumption that the potential L has representation (2.4) and is a convex function of its arguments. Therefore, we shall not dwell on this in detail. As in Sec. 1, it is assumed that:

— The stoichiometric coefficients $\nu_j^{(s)}$ of each of the reactions are subject to the condition

$$\sum_j \nu_j^{(s)} = 0; \tag{2.6}$$

— $L_{q_j} = \rho N_j$, i.e., the potentials are equal to the mass of the jth substance in unit volume $\Big(\sum_j N_j = 1\Big)$;

— $\rho = \sum_j L_{q_j}$.

Then, from (2.5) and (2.6), it follows that the mass conservation law becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}\Big(u_k \rho + \sum_j v_k^{(j)}\Big) = 0.$$

By the same line of reasoning as that used to substantiate the consistency of (2.1) with (2.3), it can be shown that (2.5) is consistent with the additional equations

$$\frac{\partial L_{v_2^{(j)}}}{\partial x_3} - \frac{\partial L_{v_3^{(j)}}}{\partial x_2} = 0, \qquad \frac{\partial L_{v_3^{(j)}}}{\partial x_1} - \frac{\partial L_{v_1^{(j)}}}{\partial x_3} = 0, \qquad \frac{\partial L_{v_1^{(j)}}}{\partial x_2} - \frac{\partial L_{v_2^{(j)}}}{\partial x_1} = 0. \tag{2.7}$$

Simultaneous fulfillment of (2.5) and (2.7) implies the validity of the conservation laws

$$\begin{gathered} \frac{\partial L_{u_i}}{\partial t} + \frac{\partial}{\partial x_k}\Big[(u_k L)_{u_i} + \sum_j (v_k^{(j)} L_{v_i^{(j)}} - \delta_{ik} v_r^{(j)} L_{v_r^{(j)}})\Big] = 0, \\ \frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_k}\Big\{u_k(\mathcal{E} + L) + \sum_j [(u_i v_k^{(j)} - u_k v_i^{(j)}) L_{v_i^{(j)}} + q_j v_k^{(j)}]\Big\} = 0, \end{gathered} \tag{2.8}$$

which model momentum and energy conservation. Here

$$\mathcal{E} = q_j L_{q_j} + u_i L_{u_i} + v_i^{(j)} L_{v_i^{(j)}} + T L_T - L.$$

In addition, using (2.7), the second line in (2.5) is also transformed to the conservation law

$$\frac{\partial L_{v_i^{(j)}}}{\partial t} + \frac{\partial}{\partial x_i}\,[(u_k L)_{v_k^{(j)}} + q_i^{(j)}] = 0. \tag{2.9}$$

Let us dwell on the inclusion of dissipative viscous terms in Eqs. (2.5).

In the equations of the first line in (2.5) and in the momentum conservation law, the zero on the right side is easily replaced by $\partial \sigma_{ik}/\partial x_k$, where

$$\sigma_{ik} = \frac{1}{2}\,\alpha\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big) + \beta \delta_{ik}\,\frac{\partial u_r}{\partial x_r}$$

($\alpha \geqslant 0$ and $\beta \geqslant 0$ are the viscosities), i.e., the momentum conservation law becomes

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial}{\partial x_k}\Big[(u_k L)_{u_i} + \sum_j (v_k^{(j)} L_{v_i^{(j)}} - \delta_{ik} v_r^{(j)} L_{v_r^{(j)}}) - \sigma_{ik}\Big] = 0.$$

However, the second line in (2.5) or in the conservation law (2.9) should be changed with caution. The fact is that the consistency of system (2.5) and the additional equations (2.7) is deduced from an analysis of the second line. It is these additional equations that allow one to establish that the nondivergent equalities in (2.5) are equivalent to the conservation laws (2.8) and (2.9). The zero right side in the second line in (2.5) can be replaced only by a vector with components $f^{(j)}$ such that $\varepsilon_{ikl}\,\partial f_l^{(j)}/\partial x_k = 0$. In this case, the proof of the consistency of (2.5) with (2.7) will not be violated. In particular, we can set

$$f_l^{(j)} = \frac{\partial}{\partial x_l}\,\hat{\sigma}^{(j)}, \qquad \hat{\sigma}^{(j)} = \gamma^{(j)}\,\frac{\partial v_k^{(j)}}{\partial x_k},$$

replacing (2.9) by

$$\frac{\partial L_{v_i^{(j)}}}{\partial t} + \frac{\partial}{\partial x_i}\,[(u_k L)_{v_k^{(j)}} + q_j] = \frac{\partial}{\partial x_i}\,\hat{\sigma}^{(j)}.$$

The inclusion of dissipative terms also leads to a certain modification of the entropy equation [the last line in (2.5)] and the energy conservation law:

$$\frac{\partial L_T}{\partial t} + \frac{\partial}{\partial x_k}\,(u_k L)_T = \frac{1}{T}\Big[\Big(\sum_l L_{q_l}\Big)\sum_s \frac{(\nu_j^{(s)} q_j)^2}{\tau_s} + \frac{\partial u_i}{\partial x_k}\,\sigma_{ik} + \sum_j \hat{\sigma}^{(j)}\,\frac{\partial v_i^{(j)}}{\partial x_i}\Big],$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_k}\,\Big\{u_k(\mathcal{E} + L) - u_i \sigma_{ik} + \sum_j [(u_i v_k^{(j)} - u_k v_i^{(j)}) L_{v_i^{(j)}} + v_k^{(j)}(q_j - \hat{\sigma}^{(j)})]\Big\} = 0.$$

Accounting for diffusion, thermal diffusion, and thermal conductivity is similar to that in Sec. 1 and does not require additional explanations.

In some cases, it may be useful to describe the thermal state of the media invoking several parameters rather than only one temperature, for example, as is done in plasma theory, where ion and electron temperatures are considered.

It may be more convenient to introduce two temperatures T and θ and assume that θ is added to the number of arguments of the potential L. In this case, in the equations for the two entropies L_T and L_θ, the heat sources for them included in the right sides must be arranged in these equations, for example, as follows:

$$\frac{\partial L_T}{\partial t} + \frac{\partial}{\partial x_k}\,(u_k L_T) = \frac{1}{T}\Big[\Big(\sum_l L_{q_l}\Big)\sum_s \frac{(\nu_j^{(s)} q_j)^2}{\tau_s} + \frac{\partial u_i}{\partial x_k}\,\sigma_{ik}\Big] + æ\,\frac{\theta - T}{T},$$

$$\frac{\partial L_\theta}{\partial t} + \frac{\partial}{\partial x_k}\,(u_k L_\theta) = \frac{1}{\theta}\Big[\sum_j \frac{\partial v_i^{(j)}}{\partial x_i}\,\hat\sigma^{(j)}\Big] + æ\,\frac{T - \theta}{\theta}.$$

Here the right sides include the additional terms $æ(\theta - T)/T$ and $æ(T - \theta)/\theta$, which do not disturb the energy balance and contribute to equalization of the temperatures T and θ. Since for $æ \geqslant 0$,

$$\frac{1}{T}\,æ(\theta - T) + \frac{1}{\theta}\,æ(T - \theta) = æ\,\frac{(T - \theta)^2}{T\theta} \geqslant 0,$$

the total entropy $L_T + L_\theta$ increases:

$$\frac{\partial\,(L_T + L_\theta)}{\partial t} + \frac{\partial}{\partial x_k}\,[u_k(L_T + L_\theta)] \geqslant 0.$$

In some cases, it may prove necessary to relate the "additional" entropy L_θ not to the velocity components u_i but to the parameters $v_i^{(j)}$ of one of the introduced internal mass flows, assuming that $q_{j_0} = \theta$. This, for example, allows one to describe the occurrence of "second sound" in superfluid helium (see [6, 12–15]). In this case, the corresponding equations are chosen in the form

$$\frac{\partial L_\theta}{\partial t} + \frac{\partial\,[(u_k L)_\theta + v_k^{(j_0)}]}{\partial x_k} = \frac{1}{\theta}\Big[\gamma^{(j_0)}\Big(\frac{\partial v_i^{(j_0)}}{\partial x_i}\Big)^2 + æ(T - \theta)\Big],$$

$$\frac{\partial}{\partial t}\,L_{v_i^{(j_0)}} + \frac{\partial}{\partial x_k}\,(u_k L)_{v_i^{(j_0)}} - L_{v_i^{(j_0)}}\,\frac{\partial u_k}{\partial x_k} + L_{v_k^{(j_0)}}\,\frac{\partial u_k}{\partial x_i} + \frac{\partial \theta}{\partial x_i} = \frac{\partial}{\partial x_i}\Big(\gamma^{(j_0)}\,\frac{\partial v_k^{(j_0)}}{\partial x_k}\Big).$$

The terms $L_\theta \equiv L_{q_{j_0}}$, of course, should not be included in the sum that defines the density ρ. At this point we finish our description of one of the possible thermodynamically consistent Galilean-invariant models of the mechanics of multiphase systems.

REFERENCES

1. S. K. Godunov, *Elements of Continuum Mechanics* [in Russian], Nauka, Moscow (1978).
2. S. K. Godunov and E. I. Romensky, *Elements of Continuum Mechanics and Conservation Laws* [in Russian], Nauchnaya Kniga, Novosibirsk (1988).
3. S. K. Godunov and E. I. Romensky, *Elements of Continuum Mechanics and Conservation Laws*, Kluwer Acad. Publ., Dordrecht (2003).
4. S. K. Godunov and V. M. Gordienko, "The simplest Galilean-invariant and thermodynamically consistent conservation laws," *J. Appl. Mech. Tech. Phys.*, **43**, No. 1, 1–12 (2002).
5. S. K. Godunov and V. M. Gordienko, "Complicated structures of Galilean-invariant conservation laws," *J. Appl. Mech. Tech. Phys.*, **43**, No. 2, 175–189 (2002).
6. L. D. Landau and E. M. Lifshits, *Course of Theoretical Physics*, Vol. 6: *Fluid Mechanics*, Pergamon Press, Oxford-Elmsford, New York (1987).
7. V. M. Teshukov and S. L. Gavrilyuk, "Kinetic model for the motion of compressible bubbles in a perfect fluid," *Europ. J. Mech. B. Fluids*, **21**, 469–491 (2002).
8. S. L. Gavrilyuk and V. M. Gavrilyuk, "Generalized vorticity for bubbly liquid and dispersive shallow water equations," *Continuum Mech. Thermodyn.*, **13**, 365–382 (2001).

9. E. I. Romensky, “Thermodynamics and hyperbolic systems of balance laws in continuum mechanics,” in: E. F. Toro (ed.), *Godunov Methods: Theory and Application*, Kluwer Acad./Plenum Publ., New York (2001), pp. 745–761.
10. Yu. B. Rymer and M. Sh. Ryvkin, *Thermodynamics, Statistical Physics, and Kinetics* [in Russian], Nauka, Moscow (1977).
11. M. A. Rydalevskaya, *Statistical and Kinetic Models in Physicochemical Gas Dynamics* [in Russian], Izd. St. Petersburg Univ., St. Petersburg (2003).
12. V. N. Dorovskii, “Equations of the continual theory of filtration,” Preprint No. 9, Institute of Geology and Geophysics, Novosibirsk (1987).
13. V. N. Dorovskii, “Continual theory of filtration,” *Geolog. Geofiz.*, No. 7, 39–45 (1989).
14. V. N. Dorovskii and Yu. V. Perepechko, “Phenomenological description of two-velocity media with relaxing shear stresses,” *J. Appl. Mech. Tech. Phys.*, No. 3, 403–409 (1992).
15. A. M. Blokhin and V. N. Dorovskii, *Problems of Mathematical Modeling in the Theory of Multivelocity Continuum* [in Russian], Joint Inst. of Geology and Geophysics, Novosibirsk (1994).